

# CHARACTER TABLES REFLECTING CORRESPONDENCES BETWEEN HOPF ALGEBRAS AND THEIR DUALS

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**ABSTRACT.** We continue studying properties of semisimple Hopf algebras  $H$  over algebraically closed fields of characteristic 0 resulting from their generalized character tables. We show that just as normal subgroups of a finite group can be recovered from its character table so does the generalized character table of  $H$  reflect the normal left coideal subalgebras of  $H$ ,  $N$ . These  $N$  are the Hopf analogues of normal subgroups in the sense that they arise from Hopf quotients. Such  $N$  are invariants of  $H$  under the left hit action of Hopf subalgebras of  $H^*$ , they appear as Hopf algebraic analogues of (semi) kernels and as images of the Drinfeld map of quasitriangular Hopf algebras. We apply these ideas to prove Hopf analogues of known results in group theory. Among the rest we prove that columns of the character table are orthogonal and that all entries are algebraic integers. We prove a Burnside-Brauer theorem for almost cocommutative  $H$ . We also prove the Hopf algebras analogue of the following (Burnside) theorem: If  $G$  is a nonabelian simple group then  $\{1\}$  is the only conjugacy class of  $G$  which has prime power order.

We dedicate this paper to the late Bettina Zoeller-Richmond whose joint seminal work with Nichols is at the basis of many of the results in this paper.

## INTRODUCTION

In this paper we mainly continue studying properties of semisimple Hopf algebras  $H$  over an algebraically closed field of characteristic 0 with special emphasis on Hopf algebraic analogues of character tables. Unlike character tables for groups they need not be square matrices, they do however reflect various properties of  $H$ . For example, an important application of character tables of finite groups  $G$  is the fact that all normal subgroups of  $G$  can be recovered from them. Analogously

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we show that for semisimple Hopf algebras  $H$ , all normal left coideal subalgebras can be recovered from their character tables.

This strengthens the basic philosophy that the analogue of normal subgroups for Hopf algebras are normal left coideal subalgebras,  $N$ , rather than the more naive notion of normal Hopf subalgebras. Such  $N$  occur whenever one deals with Hopf quotients. We show in fact that such  $N$  can be realized as invariants under the left hit actions of Hopf subalgebras of  $H^*$ . Specific examples are the so called left kernels of representations of  $H$ , which are analogues of kernels of representations of a group  $G$ , (an idea introduced by Burciou). We also introduce semi-kernels, a Hopf algebra analogue of certain normal subgroups of  $G$  which contain such kernels.

In a different direction, trivial for groups, examples of normal left coideal subalgebras are images of the Drinfeld map for quasitriangular Hopf algebras.

We apply these ideas to prove among the rest following (Burnside) theorem: If  $G$  is a nonabelian simple group then  $\{1\}$  is the only conjugacy class of  $G$  which has prime power order.

The paper is organized as follows: In the preliminaries we define conjugacy classes  $\mathfrak{C}_j$ , normalized class sums  $\eta_j$  and a generalized character table  $(\xi_{ij})$  as follows:

Let  $\{\frac{1}{d}\lambda = F_0, \dots, F_{m-1}\}$  be a complete set of central primitive idempotents of  $R(H)$ . For each  $j$ , let  $\{f_{ij}\}$ ,  $1 \leq i \leq m_j$  be a full set of primitive orthogonal idempotents of the algebra  $F_j R(H)$ . Then  $f_{ij} R(H)$  are all isomorphic as right  $R(H)$ -modules of dimension  $m_j$ . The **conjugacy class**  $\mathfrak{C}_{ij}$  is defined as:

$$\mathfrak{C}_{ij} = \Lambda \leftarrow f_{ij} H^*.$$

For each  $j$  choose a representative for each conjugacy class by taking arbitrarily  $i$ , and defining:

$$f_j = f_{ij} \quad \mathfrak{C}_j = \mathfrak{C}_{ij}.$$

Define also

$$\mathfrak{C}^j = \Lambda \leftarrow F_j H^* = \bigoplus_i \mathfrak{C}_{ij}.$$

We generalize also the notions of **Class sum** and of a representative of a conjugacy class as follows:

$$C_j = \Lambda \leftarrow dF_j, \quad \eta_j = \frac{1}{\dim(f_j H^*)} C_j.$$

We refer to  $\eta_j$  as a **normalized class sum**.

The generalized character table  $(\xi_{ij})$  of a semisimple Hopf algebra  $H$  over  $k$  is given by:

$$\xi_{ij} = \langle \chi_i, \eta_j \rangle,$$

$$0 \leq i \leq n-1, 0 \leq j \leq m-1.$$

In §2 we describe certain correspondences between  $H^*$  and  $H$ . Let  $\rightarrow$  denote the left hit action of  $H^*$  on  $H$ . For any subalgebra  $T$  of  $H^*$  we define the  $H$ -invariants under  $\rightarrow$ :

$$H^T = \{h \in H \mid b \rightarrow h = \langle b, 1 \rangle h, \forall b \in T\}.$$

We prove:

**Theorem 2.4:** Let  $H$  be a finite dimensional Hopf algebra, then there exists a bijective correspondence between left coideal subalgebras  $T$  of  $H^*$  and left coideal subalgebras  $A$  of  $H$ . The maps

$$T \rightarrow H^T \quad A \rightarrow (H^*)^A$$

are inverses of each other, that is,

$$T = (H^*)^{H^T} \quad A = H^{(H^*)^A}$$

This correspondence gives an alternative description of left-kernels of an  $H$ -representation  $V$ ,  $\text{LKer}_V$  defined by Burciu as:

$$\text{LKer}_V = \{h \in H \mid \sum h_1 \otimes h_2 \cdot v = h \otimes v, \forall v \in V\}.$$

We show:

**Theorem 2.9:** Let  $H$  be a Hopf algebra and  $V$  a finite dimensional representation of  $H$  with afforded character  $\chi_V$ . Let  $B_V$  be the bialgebra (and thus the Hopf subalgebra) of  $H^*$  generated by  $\chi_V$ . Then

$$H^{B_V} = \text{LKer}_V.$$

Just as for groups we say that  $\chi_V$  is a faithful character if  $\text{LKer}_V = k1$ . Giving an alternative proof of the generalized Burnside-Brauer theorem ([Bu2, 4.2.1]). If the character algebra  $R(H)$  is assumed to be commutative we get a complete analogue of the Burnside-Brauer theorem as follows. Define the value set

$$\chi(H) = \{\langle \chi, \eta_j \rangle \mid \text{all } \eta_j\}.$$

Then:

**Theorem 2.12:** Let  $H$  be a semisimple Hopf algebra such that  $R(H)$  is commutative and let  $\chi$  be a faithful character of  $H$ . Set

$$t = \text{number of distinct values in } \chi(H).$$

Then each character  $\mu$  of  $H$  appears with positive multiplicity in at least one of  $\{\varepsilon, \chi, \chi^2, \dots, \chi^{t-1}\}$ .

We then show:

**Theorem 2.13:** Let  $H$  be a semisimple Hopf algebra over an algebraically closed field of characteristic 0. Then the following are equivalent:

- (i) There exists a 1-dimensional representation of  $H$  with a faithful character.
- (ii)  $R(H) = k \langle \sigma \rangle$ , where  $\sigma$  is a grouplike element of  $H^*$ .
- (iii)  $H = (k \langle \sigma \rangle)^*$ ,  $\sigma$  is a grouplike element of  $H^*$ .

In §3 we prove that normalized class sums  $\eta_j$  are in fact irreducible characters of  $R(H)$ .

**Theorem 3.1:** Let  $H$  be a semisimple Hopf algebra over  $k$ ,  $\{\eta_j\}$  the normalized class sums of  $H$ . Then the irreducible character  $\mu_j$  of  $R(H)$  corresponding to the irreducible  $R(H)$ -module  $f_j R(H)$  can be identified inside  $Z(H)$  as  $\eta_j$ .

We use this to prove orthogonality of columns of the generalized character table.

**Theorem 3.5:** Let  $H$  be a  $d$ -dimensional semisimple Hopf algebra over  $\mathbb{C}$ , then

$$\sum_k \xi_{ki} \overline{\xi_{kj}} = \delta_{ij} \frac{d \dim(f_i R(H))}{\dim(f_i H^*)}.$$

Another property of the generalized character table, which is known for groups and is of essential importance in proving later the existence of various normal left coideal subalgebras, is the following:

**Theorem 3.8:** Let  $H$  be a semisimple Hopf algebra over  $\mathbb{C}$  with a character table  $(\xi_{ij})$  where  $\xi_{ij} = \langle \chi_i, \eta_j \rangle$  and  $f_j R(H)$  an irreducible right  $R(H)$  module of dimension  $m_j$ . Then

- (i) Each entry of the  $i$ -th row of the character table satisfies:

$$|\langle \chi_i, \eta_j \rangle| \leq m_j \langle \chi_i, 1 \rangle.$$

- (ii) Equality holds if and only if right multiplication by  $\chi_i$ ,  $r_{\chi_i}$  acts on  $f_j R(H)$  as  $\alpha_i \text{Id}_{f_j R(H)}$ , where  $|\alpha_i| = \langle \chi_i, 1 \rangle$ .
- (iii)  $\langle \chi_i, \eta_j \rangle = m_j \langle \chi_i, 1 \rangle$  if and only if  $r_{\chi_i}$  acts on  $f_j R(H)$  as  $\langle \chi_i, 1 \rangle \text{Id}_{f_j R(H)}$ .

We extend the definition of left kernels to  $\omega$ -left kernels,  $\omega \in \mathbb{C}$ , as follows:

$$\text{LKer}_V^\omega = \{h \in H \mid \sum h_1 \otimes h_2 \cdot v = \omega h \otimes v \quad \forall v \in V, |\omega| = 1\}.$$

Then  $\text{LKer}_V^\omega$  is a left coideal stable under the adjoint action of  $H$  and  $\sum_\omega \bigoplus \text{LKer}_V^\omega$  is a graded normal left coideal subalgebra of  $H$ . For groups this boils down to the normal subgroup of  $G$  sometimes denoted by  $Z(\chi_V)$  ([I, Def 2.26]).

We prove essential relations between the character table and  $\omega$ -left kernels by using the an equivalence relation on characters introduced by [NR2, Prop. 18].

**Theorem 3.14:** Let  $H$  be a semisimple Hopf algebra over an algebraically closed field  $k$  of characteristic 0,  $\chi_i$  an irreducible character and  $F_j$  a central primitive idempotent of  $R(H)$ . Then the following are equivalent:

- (i)  $F_j \chi_i = \omega_i \langle \chi_i, 1 \rangle F_j$ ,  $|\omega_i| = 1$ .
- (ii)  $\langle \chi_i, \eta_j \rangle = \omega_i m_j \langle \chi_i, 1 \rangle$ ,  $|\omega_i| = 1$ .
- (iii)  $\mathfrak{C}^j \subset \text{LKer}_{V_i}^{\omega_i}$ .

Moreover, if the equivalent conditions hold then  $\omega_i$  is a root of unity of order  $t$ , where  $t$  divides  $\dim H$ .

In particular, when  $\omega = 1$  we relate normal left coideal subalgebras which are left kernels to the character table:

**Theorem 3.15:** Let  $H$  be a semisimple Hopf algebra over an algebraically closed field  $k$  of characteristic 0,  $B$  the Hopf subalgebra of  $H^*$  generated by  $\chi_i$ , and  $F_j$  a central primitive idempotent of  $R(H)$ . Then the following are equivalent:

- (i)  $F_j \chi_i = \langle \chi_i, 1 \rangle F_j$
- (ii)  $\langle \chi_i, \eta_j \rangle = m_j \langle \chi_i, 1 \rangle$ .
- (iii)  $\mathfrak{C}^j \subset \text{LKer}_{V_i} = H^B$ .

In §4 we apply the results of the previous sections with known results for groups. In particular we prove a theorem of Burnside for groups in our context.

**Theorem 4.5:** Let  $H$  be a non-commutative quasitriangular semisimple Hopf algebra over  $\mathbb{C}$  whose only normal left coideal subalgebras are  $\mathbb{C}$  and  $H$ . Then  $\mathbb{C}$  is the only conjugacy class of  $H$  with a prime power dimension.

This theorem is an essential step in the proof of Burnside  $p^a q^b$  theorem for groups, which was generalized to fusion categories in [ENO]. We prove here an interesting property of normal left coideal subalgebras of factorizable Hopf algebras of dimension  $p^a q^b$ .

**Theorem 4.10:** Let  $H$  is a factorizable semisimple Hopf algebra of dimension  $p^a q^b$ ,  $a + b > 0$ , and let  $N$  be a normal left coideal subalgebra

of  $H$ . Then  $N$  contains a central grouplike element. In particular, any minimal normal left coideal subalgebra of  $H$  is a central Hopf subalgebra of  $H$  of prime order.

## 1. PRELIMINARIES

Throughout this paper,  $H$  is a finite-dimensional Hopf algebra over a field  $k$ , in some cases we assume  $k = \mathbb{C}$ . We denote by  $S$  and  $s$  the antipodes of  $H$  and  $H^*$  respectively and  $\Lambda$  and  $\lambda$  the left and right integrals of  $H$  and  $H^*$  respectively so that  $\langle \lambda, \Lambda \rangle = 1$ . We say that  $H$  is unimodular if  $\Lambda$  is also a right integral (this is the case when  $H$  is semisimple). Denote by  $Z(H)$  the center of  $H$ .

Recall that any subbialgebra of  $H$  is necessarily a Hopf subalgebra.

The Hopf algebra  $H^*$  becomes a right and left  $H$ -module by the *hit* actions  $\leftarrow$  and  $\rightarrow$  defined for all  $a \in H$ ,  $p \in H^*$ ,

$$\langle p \leftarrow a, a' \rangle = \langle p, aa' \rangle \quad \langle a \rightarrow p, a' \rangle = \langle p, a'a \rangle$$

$H$  becomes a left and right  $H^*$  module analogously.

Denote by  ${}_ad$  the left adjoint action of  $H$  on itself, that is, for all  $a, h \in H$ ,

$$h{}_ad a = \sum h_1 a S(h_2)$$

A left coideal subalgebra of  $H$  is called *normal* if it is stable under the left adjoint action of  $H$ .

Let  $D(H)$  denote the Drinfeld double of the Hopf algebra  $H$ . It is not hard to see that  $H$  is a  $D(H)$ -module with respect to the left adjoint action of  $H$  on itself and the left action  $\rightarrow$  of  $H^*$  on  $H$ , defined by

$$p \rightarrow h = h \leftarrow s^{-1}(p).$$

Denote by  $\text{Coc}(H^*)$  the algebra of cocommutative elements of  $H^*$  which equals the set of all  $p \in H^*$  so that  $\langle p, ab \rangle = \langle p, ba \rangle$  for all  $a, b \in H$ . Denote by  $R(H)$  the  $k$ -span of all irreducible characters. It is an algebra (called the character algebra) which is contained in  $\text{Coc}(H^*)$ . A necessary and sufficient for them to equal is that  $H$  is semisimple.

Recall [Sk], any left coideal subalgebra  $A$  of  $H$  contains a left integral  $\Lambda_A$ . Moreover, if  $A_1 \subset A_2$  are left coideal subalgebras then  $A_2$  is free over  $A_1$ . This implies in particular that:

**Remark 1.1.** If  $A \neq B$  are left coideal subalgebras of  $H$  then  $\Lambda_A \neq \Lambda_B$ . If  $H$  is semisimple, then  $A$  is semisimple and  $\langle \Lambda_A, 1 \rangle \neq 0$ .

Let  $V$  be a finite dimensional left  $H$ -module with a character  $\chi_V$  and let  $R_V$  be the coalgebra in  $H^*$  generated by  $\chi_V$ . Then  $R_V$  can be described explicitly as follows: Let  $\delta_V$  be the corresponding right  $H^*$ -comodule structure map on  $V$ , then for a basis  $\{v_1, \dots, v_n\}$  of  $V$ , we have  $\delta_V(v_i) = \sum_k v_k \otimes x_{ki}$ , and  $\chi_V = \sum x_{ii}$ . It follows that

$$(1) \quad \delta_V(V) \subset V \otimes R_V.$$

Note that  $R_V$  contains all the irreducible constituent of  $\chi_V$ , and thus  $V$  is irreducible if and only if  $R_V$  is a simple coalgebra. Note also that for any finite dimensional left  $H$ -modules  $V, W$ ,

$$R_V R_W = R_{V \otimes W}.$$

Let  $H$  be a semisimple Hopf algebra  $\{V_0, \dots, V_{n-1}\}$  a complete set of non-isomorphic irreducible  $H$ -modules,  $V_0 = k$ , and  $\{E_0, \dots, E_{n-1}\}$  and  $\{\chi_0, \dots, \chi_{n-1}\}$  be the associated central primitive idempotents and irreducible characters of  $H$  respectively, where  $E_0 = \Lambda$  and  $\chi_0 = \varepsilon$ . Let  $\dim V_i = d_i = \langle \chi_i, 1 \rangle$ , then  $\lambda = \chi_H = \sum_{i=0}^{n-1} d_i \chi_i$ . One has (see e.g [Sc, Cor.4.6]):

$$(2) \quad \langle \chi_i, E_j \rangle = \delta_{ij} d_j, \quad \Lambda \leftarrow \chi_j = \frac{1}{d_j} S(E_j).$$

In particular,  $\{\chi_i\}, \{\frac{1}{d_j} E_j\}$  are dual bases of  $R(H)$  and  $Z(H)$  respectively.

Since  $H$  is semisimple, we have:

$$\chi_i \chi_j = \sum_l m_{ij}^l \chi_l,$$

where  $m_{ij}^l$  are non-negative integers.

For a semisimple Hopf algebra over an algebraically closed field  $k$  of characteristic 0, let  $\text{Irr}(H)$  be the set of irreducible characters of  $H$ . Then any simple subcoalgebra  $B_i$  of  $H^*$  contains precisely one character that generates  $B_i$  as a coalgebra. Since  $B = \bigoplus_{i \in I} B_i$ , where each  $B_i$  is a simple subcoalgebra of  $H^*$ , it follows that  $B$  is the coalgebra generated by  $B \cap \text{Irr}(H)$ . Also, if  $\chi \in B$  then all its irreducible constituents belong to  $B$  as well.

In particular, if  $B$  is a Hopf subalgebra of  $H^*$  then

$$\lambda_B = \sum_{\chi_i \in \text{Irr}(H) \cap B} d_i \chi_i$$

is a nonzero integral for  $B$ .

It is known ([Kac, Z]) that when  $H$  is semisimple then so is  $R(H)$ . Let  $\{\frac{1}{d} \lambda = F_0, \dots, F_{m-1}\}$  be a complete set of central primitive idempotents

of  $R(H)$ . For each  $j$ , let  $\{f_{ij}\}$ ,  $1 \leq i \leq m_j$  be a full set of primitive orthogonal idempotents of the algebra  $F_j R(H)$ . Then  $f_{ij} R(H)$  are all isomorphic as right  $R(H)$ -modules of dimension  $m_j$ .

Motivated by group theory the **conjugacy class**  $\mathfrak{C}_{ij}$  as:

$$(3) \quad \mathfrak{C}_{ij} = \Lambda \leftarrow f_{ij} H^*.$$

Then we have shown in [CW3, Th.1.4] that:

$\mathfrak{C}_{ij}$  is an irreducible  $D(H)$ -module.

Moreover,  $\mathfrak{C}_{ij} \cong \mathfrak{C}_{tj}$  as left  $D(H)$ -modules for all  $1 \leq i, t \leq m_j$ . In particular, they have the same dimension, which equals  $\dim(f_{ij} H^*)$ . Choose arbitrarily  $i$ , and set

$$(4) \quad f_j = f_{ij} \quad \mathfrak{C}_j = \mathfrak{C}_{ij}.$$

We have shown in [CW3, Th.1.4] that,

$$(5) \quad H \cong \bigoplus_{j=0}^{m-1} \mathfrak{C}_j^{\oplus m_j}$$

as  $D(H)$ -modules.

We generalize also the notions of **Class sum** and of a representative of a conjugacy class as follows:

$$(6) \quad C_j = \Lambda \leftarrow dF_j, \quad \eta_j = \frac{1}{\dim(f_j H^*)} C_j.$$

We refer to  $\eta_j$  as a **normalized class sum**.

By general ring theory considerations (see e.g. [Lo, §3]),  $\dim f_j H^*$  does not depend on the choice of the primitive orthogonal idempotent  $f_j$  belonging to  $F_j$ , hence neither does  $\eta_j$ . Note that contrary to the group situation, one does not always have  $\eta_j \in \mathfrak{C}_j$ , however,  $\eta_j \in \bigoplus_{i=1}^{m_i} \mathfrak{C}_{ij}$ .

Since  $f_0 = \frac{1}{d}\lambda$ , it follows that  $f_0 = F_0$ ,  $m_0 = 1$ ,  $\dim f_0 H^* = 1$  and  $\eta_0 = 1$ .

When  $R(H)$  is commutative then the central primitive idempotents of  $R(H)$ ,  $\{F_j\}$  are also minimal idempotents, thus  $F_j = f_j$  and they form another basis for  $R(H)$ . Moreover, for any character  $\chi$  and primitive idempotent  $F_j$ ,

$$(7) \quad \chi F_j = \langle \chi, \eta_j \rangle F_j.$$

We can define now a generalized character table for  $H$  as follows:



**Definition 1.2.** The generalized character table  $(\xi_{ij})$  of a semisimple Hopf algebra  $H$  over  $k$  is given by:

$$\xi_{ij} = \langle \chi_i, \eta_j \rangle,$$

$$0 \leq i \leq n-1, 0 \leq j \leq m-1.$$

Note that  $\xi_{i0} = \langle \chi_i, 1 \rangle = d_i$  and  $\xi_{0j} = m_j = \dim f_j R(H)$  for all  $0 \leq i \leq n-1, 0 \leq j \leq m-1$ .

## 2. INVARIANTS UNDER THE LEFT *hit* ACTION

In this section we relate Hopf subalgebras of  $H^*$  and normal left coideal subalgebras of  $H$ . We do in fact realize them as invariants under the left *hit* action.

For any subalgebra  $T$  of  $H^*$ , denote by  $H^T$  the set of  $T$ -invariants of  $H$  under the left *hit* action. That is,

$$(8) \quad H^T = \{h \in H \mid b \rightharpoonup h = \langle b, 1 \rangle h, \forall b \in T\}$$

**Remark 2.1.** If  $H$  is semisimple and  $N$  is a left coideal subalgebra, then

$$(H^*)^N = \Lambda_N \rightharpoonup H^*$$

**Proposition 2.2.** Let  $H$  be a finite dimensional Hopf algebra over any field  $k$  and  $T$  a subalgebra of  $H^*$ . Then:

- (i)  $H^T$  is a left coideal of  $H$ .
- (ii) If  $T$  is a left or a right coideal subalgebra of  $H^*$  then  $H^T$  is a left coideal subalgebra of  $H$ .
- (iii) If  $T$  is a normal left subalgebra in  $H^*$  then  $H^T$  is a Hopf subalgebra of  $H$ .
- (iv) If  $T$  is a bialgebra then  $H^T$  is a normal left coideal subalgebra of  $H$ .
- (v). If  $T$  is a normal left coideal subalgebra of  $H^*$  then  $H^T$  is a Hopf subalgebra of  $H$ .

*Proof.* (i) We need to show that  $H^T \leftarrow H^* \subset H^T$ . Let  $b \in T, h \in H^T, p \in H^*$ , then

$$b \rightharpoonup (h \leftarrow p) = (b \rightharpoonup h) \leftarrow p = \langle b, 1 \rangle (h \leftarrow p).$$

Hence  $H^T$  is a left coideal.

- (ii) Assume  $T$  is a left coideal. Let  $b \in T, h, h' \in H^T$ , then

$$b \rightharpoonup (hh') = \sum (b_1 \rightharpoonup h)(b_2 \rightharpoonup h') = (b \rightharpoonup h)h' = \langle b, 1 \rangle hh'.$$

The second equality follows from the fact that  $T$  is a left coideal. The same proof works if  $T$  is a right coideal subalgebra.

(iii). By (ii) all we need to show is that  $H^T$  is a right coideal, that is  $H^* \rightharpoonup H^T \subset H^T$ . Since  $T$  is stable under the left adjoint action, we have that

$$bp = \sum p_3(s^{-1}(p_2)bp_1) = \sum p_2(s^{-1}(p_1)_{ad}b) \in H^*T$$

for all  $b \in T$ ,  $p \in H^*$ . Hence we have for  $b \in T$ ,  $h \in H^T$ ,  $p \in H^*$ ,

$$b \rightharpoonup (p \rightharpoonup h) = bp \rightharpoonup h = \sum p_3s^{-1}(p_2bp_1) \rightharpoonup h = \langle b, 1 \rangle (p \rightharpoonup h).$$

So  $(p \rightharpoonup h) \in H^T$ .

(iv). Assume  $T$  is a coalgebra. We need to show that  $H^T$  is normal in  $H$ . Let  $h \in H^T$ ,  $x \in H$ ,  $b \in T$ , then

$$\begin{aligned} b \rightharpoonup \sum x_1hS(x_2) &= \\ &= \sum (b_1 \rightharpoonup x_1)(b_2 \rightharpoonup h)(b_3 \rightharpoonup S(x_2)) \\ &= \sum (b_1 \rightharpoonup x_1)h(b_2 \rightharpoonup S(x_2)) \quad (\text{since } B \text{ is a coalgebra}) \\ &= \sum \langle b_1, x_2 \rangle \langle b_2, S(x_3) \rangle x_1hS(x_4) \\ &= \sum \langle b, 1 \rangle x_1hS(x_2). \end{aligned}$$

Hence  $\sum x_1hS(x_2) \in H^T$ .

(v). By (ii) and (iii)  $H^T$  is a bialgebra. Since  $H$  is finite dimensional it follows that it is a Hopf subalgebra.  $\square$

We have,

**Lemma 2.3.** *For any subalgebra  $T$  of  $H^*$  and a left coideal  $A$  of  $H$  we have:*

$$(9) \quad A \subset H^T \Leftrightarrow \langle p, a \rangle = \langle p, 1 \rangle \langle \varepsilon, a \rangle \quad \forall a \in A, p \in T.$$

*If  $T$  and  $A$  are left coideal subalgebras of  $H^*$  and  $H$  respectively, then*

$$(10) \quad A \subset H^T \Leftrightarrow T \subset (H^*)^A.$$

*In particular,*

$$(11) \quad T \subset (H^*)^{H^T} \quad A \subset H^{(H^*)^A}$$

*Proof.* By applying  $\varepsilon \otimes \text{Id}$  to both sides of (8) we get the right hand side of the equivalence in (9). Conversely, if  $A$  is a left coideal satisfying the right hand side of the equivalence, then  $A \subset H^T$  by the definition of  $H^T$ .

(10) follows directly from (9).

The first inclusion in (11) follows by taking  $A = H^T$  in (10). The second inclusion follows by replacing  $H$  with  $H^*$  and taking  $T = (H^*)^A$ .  $\square$

We can show now,

**Theorem 2.4.** *Let  $H$  be a finite dimensional Hopf algebra, then there exists a bijective correspondence between left coideal subalgebras  $T$  of  $H^*$  and left coideal subalgebras  $A$  of  $H$ . The maps*

$$T \rightarrow H^T \quad A \rightarrow (H^*)^A$$

*are inverses of each other, that is,*

$$T = (H^*)^{H^T} \quad A = H^{(H^*)^A}$$

*Proof.* Let  $T$  be a left coideal subalgebra of  $H^*$ . Set  $A = H^T$ , then  $A$  is a left coideal subalgebra of  $H$  by Proposition 2.2(ii). By (11),  $T \subset (H^*)^A$ . By Remark 1.1, equality will follow once we prove that every nonzero left integral of  $T$  is indeed a left integral of  $(H^*)^A$ . Let  $\lambda_T$  be a left integral for  $T$ . For any  $p \in (H^*)^A$ ,

$$p\lambda_T \rightharpoonup \Lambda = p \rightharpoonup (\lambda_T \rightharpoonup \Lambda) = \langle p, 1 \rangle (\lambda_T \rightharpoonup \Lambda)$$

The last equality follows from the fact that  $(\lambda_T \rightharpoonup H) \subset H^T = A$ , and  $A \subset H^{(H^*)^A}$  by (11). hence  $p\lambda_T = \langle p, 1 \rangle \lambda_T$  and we are done. The proof that  $A = H^{(H^*)^T}$  is identical replacing  $H$  by  $H^*$  and  $T$  by  $A$  in the above arguments.  $\square$

It follows from the theorem above and Proposition 2.2(iii) that we can relate in particular normal left coideal subalgebras of  $H$  and Hopf subalgebras of  $H^*$ .

**Corollary 2.5.** *Let  $H$  be a finite dimensional Hopf algebra. Then there exists a bijective correspondence between Hopf subalgebras  $B$  of  $H^*$  and normal left coideal subalgebras  $N$  of  $H$  given by:*

$$B \rightarrow H^B, \quad N \rightarrow (H^*)^N.$$

**Remark 2.6.** The corollary above is in fact the bijective correspondence between normal left coideal subalgebras and Hopf quotients of  $H$  discussed in [Ta]. Explicitly, if  $B$  is a Hopf subalgebra of  $H^*$  and  $\pi : H \rightarrow B^*$  is the corresponding Hopf projection, that is,

$$\langle \pi(h), b \rangle = \langle h, \pi(b) \rangle,$$

then

$$H^{co\pi} = \{h \in H \mid h_1 \otimes \pi(h_2) = h \otimes 1\}.$$

Now, for all  $p \in H^*$ ,  $b \in B$ ,

$$\left\langle \sum h_1 \otimes \pi(h_2), p \otimes b \right\rangle = \langle p, b \rightharpoonup h \rangle.$$

Hence it is easy to see that

$$(12) \quad H^B = H^{co\pi}.$$

**Corollary 2.7.** Let  $N = H^B$ . Since

$$H \cong H^{co\pi} \otimes H/HN^+ \cong H^B \otimes B^*,$$

we obtain:

$$\dim(H^B) = \frac{\dim H}{\dim B}.$$

Certain normal left coideal subalgebras appear in [Bu2] as a natural generalization of kernels of group representations. For an  $H$ -module  $V$  define its left kernel as follows:

$$(13) \quad \text{LKer}_V = \{h \in H \mid \sum h_1 \otimes h_2 \cdot v = h \otimes v, \forall v \in V\}.$$

**Remark 2.8.** (i) As was noted in [Bu2], for  $H = kG$ ,  $G$  a group and  $V$  a representation of  $G$ ,

$$\text{LKer}_V = k[\text{Ker}_G V].$$

Moreover,  $\text{LKer}_V$  is a normal left coideal subalgebra of  $H$ .

(ii) Note that for finite dimensional Hopf algebra  $H$ ,  $\text{LKer}_V$  is a  $D(H)$ -module, in fact it is a  $D(H)$ -module algebra. Moreover,

$$(14) \quad \text{LKer}_V = H \Leftrightarrow \chi_V = \varepsilon.$$

(iii) If  $A$  a left coideal of  $H$  then:

$$A \subset \text{LKer}_V \Leftrightarrow a \cdot v = \langle \varepsilon, a \rangle v, \forall a \in A, v \in V.$$

This follows as in the proof of (9).

(iv) If  $V = \bigoplus_i V_i^{\otimes m_i}$  then  $\text{LKer}_V = \bigcap \text{LKer}_{V_i}$ .

As a corollary we suggest an additional description of left kernels. This can be proved directly by Proposition 2.2(iii) and Theorem 2.4. It follows however from results of [Bu2] after adaptation.

**Theorem 2.9.** *Let  $H$  be a Hopf algebra and  $V$  a finite dimensional representation of  $H$  with afforded character  $\chi_V$ . Let  $B_V$  be the bialgebra (and thus the Hopf subalgebra) of  $H^*$  generated by  $\chi_V$ . Then*

$$H^{B_V} = \text{LKer}_V.$$

*Proof.* Observe that  $B_V^\perp = \bigcap_m \text{ann}_H V^{\otimes m}$ . By [Bu2, Th. 2.3.6],  $H^{co\pi} = \text{LKer}_V$ , where  $\pi : H \rightarrow H/(\bigcap_m \text{ann}_H V^{\otimes m})$  is the canonical projection. The result follows now from (12).  $\square$

As a immediate consequence we obtain an alternative proof of the generalized Burnside-Brauer theorem given in [Bu2, 4.2.1]. We generalize the notion of faithful character to Hopf algebras as follows:

**Definition 2.10.** A character  $\chi$  associated with a representation  $V$  will be called *faithful* if  $\text{LKer}_V = k1$ .

**Theorem 2.11.** [Bu2, 4.2.1]. *Let  $H$  be a semisimple Hopf algebra over an algebraically closed field  $k$  of characteristic 0,  $V$  a representation of  $H$  with associated character  $\chi$  and  $N = \text{LKer}_V$ . Then the irreducible characters of  $\overline{H} = H/(N^+H)$  are precisely all the irreducible constituents of  $\chi^n$ ,  $n \geq 0$ .*

*Proof.* Since  $V$  is a representation of  $\overline{H}$ , with a faithful character  $\chi_V$  we may assume without loss of generality that  $\chi_V$  is faithful and  $\overline{H} = H$ . By Th.2.9,

$$H^{B_V} = \text{LKer}_V = k1 = H^{H^*},$$

where  $B_V$  is the Hopf subalgebra of  $H^*$  generated by  $\chi_V$ . By the bijective correspondence described in Theorem 2.4,  $B_V = H^*$ . This implies that each irreducible character of  $H$  appears as a constituent of some power of  $\chi$ .  $\square$

Best results are achieved when the semisimple Hopf algebra  $H$  is an almost cocommutative Hopf algebra, or equivalently, if it has a commutative character algebra (see [CW4]). In this case we give in the next theorem an explicit bound to the powers of  $\chi$  appearing above.

Define the value set

$$\chi(H) = \{\langle \chi, \eta_j \rangle \mid \text{all } \eta_j\}.$$

We can prove now a **full** analogue of Burnside-Brauer theorem:

**Theorem 2.12.** *Let  $H$  be a semisimple Hopf algebra such that  $R(H)$  is commutative and let  $\chi$  be a faithful character of  $H$ . Set*

$$t = \text{number of distinct values in } \chi(H).$$

*Then each character  $\mu$  of  $H$  appears with positive multiplicity in at least one of  $\{\varepsilon, \chi, \chi^2, \dots, \chi^{t-1}\}$ .*

*Proof.* Since  $\chi$  is faithful we have by Theorem 2.11 that all irreducible characters of  $H$  appear as constituents of powers of  $\chi$ . We will show that the  $k$ -span of  $\{\varepsilon, \chi, \chi^2, \dots, \chi^{t-1}\}$  already contains all powers of  $\chi$ . Let  $\{\alpha_0, \dots, \alpha_{t-1}\}$  be the different values of  $\langle \chi, \eta_j \rangle$  where  $\alpha_0 = \langle \chi, 1 \rangle$ . Let

$$T_i = \sum_j F_{i_j},$$

where by (7),  $\{F_{i_j}\}$  are all the central idempotents satisfying  $\chi F_{i_j} = \alpha_i F_{i_j}$ . By assumption  $T_0 = d^{-1}\lambda$ . By [CW2] we have:

$$\chi = \sum_{i=0}^{t-1} \alpha_i T_i$$

Since  $\{T_i\}$  are orthogonal idempotents we get

$$\chi^l = \sum_{i=0}^{t-1} \alpha_i^l T_i$$

for all  $0 \leq l \leq t-1$ . The Vandermonde matrix  $(\alpha_i^l)$  is invertible, hence each  $T_i$  is a linear combination of  $\{\varepsilon, \chi, \chi^2, \dots, \chi^{t-1}\}$ . Since any power of  $\chi$  is a linear combination of the  $T_i$ 's, it follows that the algebra generated by  $\chi$  inside  $R(H)$  is spanned over  $k$  by  $\{\varepsilon, \chi, \chi^2, \dots, \chi^{t-1}\}$ .  $\square$

We show next that the only Hopf algebras for which there exist a 1-dimensional representation with a faithful character satisfy a stronger condition than being almost cocommutative. They are in fact both commutative and cocommutative.

**Theorem 2.13.** *Let  $H$  be a semisimple Hopf algebra over an algebraically closed field of characteristic 0. Then the following are equivalent:*

- (i) *There exists a 1-dimensional representation of  $H$  with a faithful character.*
- (ii)  *$R(H) = k \langle \sigma \rangle$ , where  $\sigma$  is a grouplike element of  $H^*$ .*
- (iii)  *$H = (k \langle \sigma \rangle)^*$ ,  $\sigma$  is a grouplike element of  $H^*$ .*

*Proof.* If  $H = k1$  then we are done. So, assume  $H \neq k1$ .

(i) $\Rightarrow$ (ii). Since  $V$  is 1-dimensional,  $\sigma = \chi_V$  is a grouplike element and  $\sigma \neq \varepsilon$  by (14). By Theorem 2.11, each irreducible character  $\chi$  is a constituent of  $\sigma^j$ ,  $j \geq 0$ . Since  $\sigma^j$  is an irreducible character it follows that  $\chi = \sigma^j$ .

(ii) $\Rightarrow$ (iii). Since all irreducible representations of  $H$  are 1-dimensional it follows that  $H$  is a commutative algebra, hence  $H^*$  is cocommutative, hence  $H^* = R(H)$ .

(iii) $\Rightarrow$ (i). Take  $V = k\sigma$ , then  $V$  is an  $H$ -module under left *hit* and  $\chi_V = \sigma$ . By assumption  $B_V = H^*$ . By Theorem 2.9,

$$\text{LKer}_V = H^{B_V} = H^{H^*} = k1.$$

$\square$

### 3. RETRIEVING STRUCTURE PROPERTIES OF $H$ FROM ITS CHARACTER TABLE

Let  $H$  be a semisimple Hopf algebra over  $\mathbb{C}$ . Nichols and Richmond [NR1, NR2] discussed various properties of  $R(H)$  as an inner product space with involution. Their method is described as follows. For  $u = \sum \alpha_i \chi_i$ ,  $v = \sum \beta_j \chi_j$ , set

$$(15) \quad (u, v) = \sum \alpha_i \bar{\beta}_i$$

Then  $(, )$  is an inner product on  $R(H)$ . Define an involution  $*$  on characters by

$$\chi^* = s(\chi)$$

and extend it to  $R(H)$  as follows. For  $u = \sum \alpha_i \chi_i \in R(H)$  set

$$u^* = \sum \bar{\alpha}_i \chi_i^*$$

As an algebra,  $R(H)$  has its own set of characters. By [NR2, Remark 11], for all  $x \in R(H)$ ,  $\mu$  a character defined on  $R(H)$  we have:

$$(16) \quad \langle \mu, x^* \rangle = \overline{\langle \mu, x \rangle}.$$

By (2),  $Z(H)$  can be considered as a dual vector space to  $R(H)$ , but the connection is much deeper.

Recall, as a consequence of the orthogonality of characters given in [La],  $R(H)$  is a symmetric algebra with a symmetric form  $\beta$  defined by

$$\beta(p, q) = \langle \Lambda, pq \rangle,$$

where  $\Lambda$  is the corresponding central form and the Casimir element is

$$(17) \quad \sum_{k=0}^{n-1} \chi_k \otimes s(\chi_k).$$

It follows that for all  $p \in R(H)$ ,

$$(18) \quad p = \sum_k \langle \Lambda, p\chi_k \rangle s(\chi_k)$$

For  $x \in R(H)$ , let  $l_x(r_x)$  denote the operator of left (right) multiplication of  $R(H)$  by  $x$ .

**Theorem 3.1.** *Let  $H$  be a semisimple Hopf algebra over an algebraically closed field of characteristic 0,  $\{\eta_j\}$  the normalized class sums of  $H$  as defined in (6). Then the irreducible character  $\mu_j$  of  $R(H)$  corresponding to the irreducible right  $R(H)$ -module  $f_j R(H)$  can be identified inside  $Z(H)$  as  $\eta_j$ . In particular  $\langle \varepsilon, \eta_j \rangle = m_j$ .*

*Proof.* It can be seen that for all  $0 \leq j \leq m$ ,

$$(19) \quad \sum_{k=0}^{n-1} \chi_k f_j s(\chi_k) = \frac{d}{\dim(f_j H^*)} F_j.$$

See e.g. [CW2, Prop.2.1] where  $\sum_{k=0}^{n-1} \chi_k f_j s(\chi_k)$  is denoted there by  $\tau_C(f_j)$ . By the definition of  $\mu_j$  we have:

$$\langle \mu_j, x \rangle = \text{Trace}(r_x \circ l_{f_j}).$$

By the trace formula for symmetric algebras (see e.g. [CW2, Lemma 1.3]), we have:

$$\langle \mu_j, x \rangle = \text{Trace}(r_x \circ l_{f_j}) = \sum_k \langle \Lambda, \chi_k f_j s(\chi_k) x \rangle = \frac{d}{\dim(f_j H^*)} \langle \Lambda, F_j x \rangle.$$

The result follows now from the definition of  $\eta_j$ .  $\square$

**Remark 3.2.** Since  $f_j R(H) \cong f_{j'} R(H)$  for any other primitive idempotent  $f_{j'}$  of  $R(H)$  belonging to  $F_j$ , it follows that  $\mu_j$  does not depend on the choice of the primitive idempotent belonging to  $F_j$  as was demonstrated in Theorem 3.1.

As a corollary of Theorem 3.1 we obtain,

**Corollary 3.3.** All entries of the character table are algebraic integers.

*Proof.* As proved in [Lo], any character  $\chi_i$  satisfies a monic polynomial over  $\mathbb{Z}$ , hence so does its image under the representation  $\rho_{f_j R(H)}$ . It follows that all eigenvalues of  $\rho_{f_j R(H)}(\chi_i)$  are algebraic integers and hence so is  $\text{trace}(\rho_{f_j R(H)}(\chi_i))$ . By Theorem 3.1  $\text{trace}(\rho_{f_j R(H)}(\chi_i)) = \langle \chi_i, \eta_j \rangle$ .  $\square$

As a corollary of Theorem 3.1 and (16) we obtain:

**Corollary 3.4.** When  $k = \mathbb{C}$ , then for any normalized class sum in  $H$ , and for any element  $x \in R(H)$ ,

$$\langle x^*, \eta_j \rangle = \overline{\langle x, \eta_j \rangle}.$$

We denote  $\chi_i^* = s(\chi_i)$  by  $\chi_{i^*}$ . Note that since  $\Lambda$  is cocommutative, we have  $S(C_j) = S(\Lambda \leftarrow dF_j) = \Lambda \leftarrow dSF_j$ . Since  $\dim(F_j H^*) = \dim(H^* F_j) = \dim((SF_j) H^*)$ , it follows that  $S\eta_j$  is a normalized class sum as well. Denote also  $S\eta_j$  by  $\eta_{j^*}$ . Then Corollary 3.4 implies in particular that

$$(20) \quad \xi_{i^*j} = \xi_{ij^*} = \overline{\xi_{ij}}.$$

We can prove orthogonality relations for the columns of the generalized character table as follows:



**Theorem 3.5.** *Let  $H$  be a  $d$ -dimensional semisimple Hopf algebra over  $\mathbb{C}$ , then*

$$\sum_k \xi_{ki} \overline{\xi_{kj}} = \delta_{ij} \frac{d \dim(f_i R(H))}{\dim(f_i H^*)}.$$

*Proof.*

$$\begin{aligned} \sum_k \xi_{ki} \overline{\xi_{kj}} &= \\ &= \sum_k \langle \chi_k, \eta_i \rangle \langle s(\chi_k), \eta_j \rangle \quad (\text{by (20)}) \\ &= \frac{1}{\dim(H^* f_i)} \langle \sum_k \langle \chi_k, (\Lambda \leftarrow dF_i) \rangle s(\chi_k), \eta_j \rangle \\ &= \frac{d}{\dim(H^* f_i)} \langle \sum_k \langle F_i \chi_k, \Lambda \rangle s(\chi_k), \eta_j \rangle \\ &= \frac{d}{\dim(H^* f_i)} \langle F_i, \frac{1}{\dim(H^* f_j)} (\Lambda \leftarrow dF_j) \rangle \quad (\text{by (18)}) \\ &= \delta_{ij} \frac{d^2}{(\dim(f_i H^*))^2} \langle \Lambda, F_i \rangle \\ &= \delta_{ij} \frac{d \dim(f_i R(H))}{\dim(H^* f_i)} \quad (\text{by [CW1, (8) and (9)]}) \end{aligned}$$

□

**Remark 3.6.** When  $R(H)$  is commutative then  $\dim f_j R(H) = 1$ , hence for  $i = j$  the right hand side of the orthogonality relations boils down to  $\frac{d}{\dim \mathfrak{C}_j}$ . Thus, as for groups, the dimension of each conjugacy class can be recovered from the character table.

Another norm of an element  $x \in R(H)$  was defined in [NR2] as follows:

$$(21) \quad N(x) = \sup_{0 \neq u \in R(H)} \frac{\|ux\|}{\|u\|}$$

where  $\| \cdot \|$  is the norm obtained from the inner product given in (15). It was proved there [NR2, Th.14.2] that for a character  $\chi$ ,

$$\langle \chi, 1 \rangle = N(\chi)$$

This implies that if  $u \in R(H)$  is a nonzero eigenvector of  $r_\chi$  with an eigenvalue  $\alpha$ , then

$$(22) \quad |\alpha| \leq \langle \chi, 1 \rangle.$$

The following lemma about complex numbers will be an essential tool in analyzing values related to the character table.

**Lemma 3.7.** *Let  $\{\alpha_k\}_{k=1}^m$  be a set of complex numbers with  $\alpha_1 \neq 0$  and let  $\{a_k\}_{k=1}^m$  be a set of non-negative real numbers satisfying:*

$$\left| \sum_{k=1}^m \alpha_k \right| = \sum_{k=1}^m a_k \quad \text{and} \quad |\alpha_k| \leq a_k$$

for all  $1 \leq k \leq m$ . Then:

(i) *There exists an  $\omega \in \mathbb{C}$ ,  $|\omega| = 1$  so that for all  $1 \leq k \leq m$ ,*

$$\alpha_k = \omega a_k.$$

(ii). *If  $\alpha_j$  is real for some  $1 \leq j \leq m$ , then  $\alpha_k = a_k$  (thus is real) for all  $k$ .*

*Proof.* By assumptions,

$$\sum_{k=1}^m a_k = \left| \sum_{k=1}^m \alpha_k \right| \leq \sum_{k=1}^m |\alpha_k| \leq \sum_{k=1}^m a_k.$$

Hence both inequalities are equalities. The last equality and the second assumption implies that  $|\alpha_k| = a_k$  or equivalently,  $\alpha_k = \omega_k a_k$  where  $|\omega_k| = 1$ . We wish to show that  $\omega_k = \omega_1$  for all  $k$ .

Well, the triangle inequality for complex numbers implies that for any  $k$  there exists a non-negative real number  $r_k$  so that  $\alpha_k = r_k \alpha_1$ , that is,

$$(23) \quad \omega_k a_k = r_k \omega_1 a_1.$$

Since  $|\omega_k| = |\omega_1| = 1$ , we have,  $r_k = \frac{a_k}{a_1}$ . Substituting in (23), we get  $\omega_k = \omega_1$  for all  $1 \leq k \leq m$ .

(ii) If  $\alpha_j$  is real for some  $j$ , then  $\omega_j = 1$  hence  $\omega_1 = 1$  and all  $\omega_k$  equal 1 as well and the result follows.  $\square$

We show:

**Theorem 3.8.** *Let  $H$  be a semisimple Hopf algebra over  $\mathbb{C}$  with a character table  $(\xi_{ij})$  where  $\xi_{ij} = \langle \chi_i, \eta_j \rangle$  and  $f_j R(H)$  an irreducible right  $R(H)$  module of dimension  $m_j$ . Then*

(i) *In each entry of the  $i$ -th row of the character table we have:*

$$|\langle \chi_i, \eta_j \rangle| \leq m_j \langle \chi_i, 1 \rangle.$$

(ii) *Equality holds if and only if  $r_{\chi_i}$  acts on  $f_j R(H)$  as  $\alpha_i \text{Id}_{f_j R(H)}$ , where  $|\alpha_i| = \langle \chi_i, 1 \rangle$ .*

(iii)  *$\langle \chi_i, \eta_j \rangle = m_j \langle \chi_i, 1 \rangle$  if and only if  $r_{\chi_i}$  acts on  $f_j R(H)$  as  $\langle \chi_i, 1 \rangle \text{Id}_{f_j R(H)}$ .*

*Proof.* (i) By (22) each eigenvalue  $\alpha_k$ , of the action of  $r_{\chi_i}$  on  $f_j R(H)$ , satisfies  $|\alpha_k| \leq \langle \chi_i, 1 \rangle$ , for all  $1 \leq k \leq m_j$ . By Theorem 3.1,  $\langle \chi_i, \eta_j \rangle$  is the trace of this action, hence,

$$|\langle \chi_i, \eta_j \rangle| = \left| \sum_{k=1}^{m_j} \alpha_k \right| \leq \sum_{k=1}^{m_j} |\alpha_k| \leq m_j \langle \chi_i, 1 \rangle.$$

(ii) If equality holds in (i), then by applying Lemma 3.7 with  $m = m_j$  and  $a_k = \langle \chi_i, 1 \rangle$  for all  $k$ , we obtain that all  $\alpha_k$  are equal and  $|\alpha_k| = \langle \chi_i, 1 \rangle$ . Hence the result follows. The converse is obvious.

(iii) If  $\langle \chi_i, \eta_j \rangle = m_j \langle \chi_i, 1 \rangle$  then by part (ii) it follows that  $r_{\chi_i}$  acts on  $f_j R(H)$  as  $\alpha \text{Id}_{f_j R(H)}$ . Thus by Theorem 3.1,  $\langle \chi_i, \eta_j \rangle = \alpha m_j$ , hence by assumption  $\alpha = \langle \chi_i, \eta_j \rangle$ . The converse is obvious again.  $\square$

**Remark 3.9.** Observe that the theorem above is independent on the choice of  $f_j \in F_j R(H)$ . This follows for part (i) from Remark 3.2 and for parts (ii) and (iii) from the right  $R(H)$ -module isomorphism between  $f_j R(H)$  and  $f_{j'} R(H)$  for any other primitive idempotent on

For a central idempotent  $F_j$  of  $R(H)$  set

$$(24) \quad I_{F_j} = \{\chi | \chi F_j = \omega_\chi \langle \chi, 1 \rangle F_j, \omega_\chi \in \mathbb{C}, |\omega_\chi| = 1\},$$

where  $\chi$  is any character of  $H$ . Based on Theorem 3.8 we obtain the following:

**Proposition 3.10.** *Let  $H$  be a semisimple Hopf algebra over  $\mathbb{C}$ ,  $F_j$  be a central primitive idempotent of  $R(H)$ ,  $f_j$  any primitive idempotent of  $R(H)$  so that  $f_j F_j = f_j$  and  $\dim f_j R(H) = m_j$ . Then:*

(i) *For any irreducible character  $\chi_i$ ,*

$$\chi_i \in I_{F_j} \Leftrightarrow |\langle \chi_i, \eta_j \rangle| = m_j \langle \chi_i, 1 \rangle.$$

(ii) *If  $\chi \in I_{F_j}$  then so does each of its irreducible constituents  $\chi_k$ , and  $\omega_{\chi_k} = \omega_\chi$ .*

(iii) *For each  $\chi \in I_{F_j}$ ,  $\omega_\chi$  is a root of unity.*

*Proof.* (i) Assume  $\chi_i \in I_{F_j}$ . Set  $\alpha_i = \omega_{\chi_i} \langle \chi_i, 1 \rangle$ . Since  $\chi_i F_j = \alpha_i F_j$  it follows that for all  $p \in R(H)$ ,

$$f_j p \chi_i = f_j p F_j \chi_i = \alpha_i f_j p.$$

That is,  $r_{\chi_i}$  acts on  $f_j R(H)$  as  $\alpha_i \text{Id}$ . This implies by Theorem 3.8(ii) that  $|\langle \chi_i, \eta_j \rangle| = m_j \langle \chi_i, 1 \rangle$ .

Conversely, if  $|\langle \chi_i, \eta_j \rangle| = m_j \langle \chi_i, 1 \rangle$  then Theorem 3.8(ii) and Remark 3.9 imply that  $r_{\chi_i}$  acts on  $f_j R(H)$  as  $\alpha_i \text{Id}$  for all primitive idempotents  $f_j$  of  $R(H)$  belonging to  $F_j$ . Since  $F_j$  is their sum it follows that  $i \in I_{F_j}$ .

(ii) if  $\chi \in I_{F_j}$  then as in the proof of (i),  $r_\chi$  acts on  $f_j R(H)$  as  $\omega_\chi \langle \chi, 1 \rangle \text{Id}$ . By Theorem 3.1,  $\langle \eta_j, \chi \rangle$  is the trace of this action, hence

$$(25) \quad \langle \chi, \eta_j \rangle = m_j \omega_\chi \langle \chi, 1 \rangle.$$

Assume  $\chi = \sum_{k=1}^n n_k \chi_k$ ,  $n_k > 0$ , then by (25),

$$m_j \sum_{k=1}^n n_k \langle \chi_k, 1 \rangle = m_j \langle \chi, 1 \rangle = m_j |\omega_\chi \langle \chi, 1 \rangle| = |\langle \chi, \eta_j \rangle| = \left| \sum_{k=1}^n n_k \langle \chi_k, \eta_j \rangle \right|.$$

Apply Lemma 3.7 with  $\alpha_k = n_k \langle \chi_k, \eta_j \rangle$  and  $a_k = n_k m_j \langle \chi_k, 1 \rangle$ , to get  $\langle \chi_k, \eta_j \rangle = \omega_\chi m_j \langle \chi_k, 1 \rangle$  for all  $1 \leq k \leq n$ . By part (i),  $\chi_k \in I_{F_j}$  and since  $\chi = \sum_{k=1}^n n_k \chi_k$  it follows that  $\omega = \omega_\chi$ .

(iii) Take  $\chi \in I_{F_j}$ . By [NR2, Prop. 5(3)],  $\varepsilon$  is a constituent of  $\chi^n$  for some  $n \geq 1$ . By part (ii),  $\omega_{\chi^n} = \omega_\varepsilon = 1$ . But  $\omega_{\chi^n} = (\omega_\chi)^n$  hence  $\omega_\chi$  is a root of unity.  $\square$

Let  $B_0$  be a Hopf subalgebra of a semisimple Hopf algebra  $B$ . In [NR2, Prop.18] the following equivalence relation was defined on simple coalgebras of  $H^*$  :

$$(26) \quad C_k \equiv_{B_0} C_{k'} \Leftrightarrow B_0 C_k \subset C_{k'}.$$

Let

$$B_k = \bigoplus_{C_{k'} \equiv_{B_0} C_k} C_{k'}.$$

Then  $B_0 B_k = B_k$  and  $B = \bigoplus_k B_k$ . By the proof of [NR2, Prop.18] one can check that the equivalence relation on simple subcoalgebras is equivalent to the following equivalence relation defined on irreducible characters inside  $B$  :

$$(27) \quad \chi_i \equiv_{B_0} \chi_j \Leftrightarrow \lambda_{B_0} \frac{\chi_i}{d_i} = \lambda_{B_0} \frac{\chi_j}{d_j}$$

Denote by  $[\chi_k]$  the equivalence class of  $\chi_k$  and set  $y_k$

$$(28) \quad y_k = \sum_{\chi_{k'} \in [\chi_k]} \langle \chi_{k'}, 1 \rangle \chi_{k'}.$$

Then  $y_0 = \lambda_{B_0}$  and  $\sum_k y_k = \lambda_B$  is an integral for  $B$  and for each  $k$ ,  $\langle y_k, 1 \rangle = \dim B_k$ . We are ready to prove the following useful proposition:

**Proposition 3.11.** *Let  $H$  be a semisimple Hopf algebra over an algebraically closed field  $k$  of characteristic 0,  $\chi$  an irreducible character and  $F_j$  a central primitive idempotent of  $R(H)$ . Let  $B$  be the Hopf algebra generated by  $\chi$ . Assume  $\chi \in I_{F_j}$ . Then the following hold:*

(i) *All irreducible characters inside  $B$  belong to  $I_{F_j}$ .*

(ii) Let  $B_0$  be the coalgebra generated by all irreducible characters  $\chi_k \in B$  so that  $F_j \chi_k = \langle \chi_k, 1 \rangle F_j$ . Then  $B_0$  is a Hopf subalgebra of  $B$ .

(iii) Inside  $B$ , consider the equivalence relation defined in (26). Set  $\omega = \omega_\chi$ . We can arrange  $B_m$  so that  $\omega_{\chi_k} = \omega^m$  for all irreducible characters  $\chi_k \in B_m$ .

(iv)  $B = \bigoplus_{m=0}^{t-1} B_m$  is a graded algebra and  $\omega$  is a root of unity. The number  $t$  is the least  $n$  so that  $\varepsilon$  is a constituent of  $\chi$ .

(v)  $\dim B_m = \dim B_0$  for all  $0 \leq m \leq t-1$ .

*Proof.* (i). Each irreducible character inside  $B$  is a constituent of  $\chi^l$  for some  $l \geq 0$ . But  $\chi^l \in I_{F_j}$  where  $\omega_{\chi^l} = \omega^l$  hence the result follows from Proposition 3.10(ii).

(ii). Since  $\omega_\chi \omega_{\chi'} = \omega_{\chi\chi'}$  for all  $\chi, \chi' \in B$ , it follows by Proposition 3.10(ii) that  $B_0$  is an algebra, and thus a Hopf subalgebra.

(iii). First note that for all  $\chi_k, \chi_{k'} \in B_k$ , we have by (27)  $\omega_{\chi_k} = \omega_{\chi_{k'}}$ . We wish to show that to each  $B_k$  corresponds a unique  $\omega_k$ . That is, if  $\chi_k \in B_k$ ,  $\chi_t \in B_t$  and  $\omega_{\chi_k} = \omega_{\chi_t}$ , then  $B_k = B_t$ .

Since  $\varepsilon$  is a constituent of  $\chi_t s(\chi_t)$  it follows that  $\omega_{s(\chi_t)} = \omega_{\chi_t}^{-1}$ . Thus  $\chi_k s(\chi_t) \in B_0$ . But then

$$\chi_k s(\chi_t) \chi_t \in B_0 \chi_t \subset B_t.$$

On the other hand, since  $\varepsilon$  is a constituent of  $s(\chi_t) \chi_t$  it follows that  $\chi_k$  is a constituent of  $\chi_k s(\chi_t) \chi_t \in B_t$ . Since  $B_t$  is a sub coalgebra it follows that  $\chi_k \in B_t$ . Thus  $B_k = B_t$ .

(iv). This follows now after arranging the subalgebras  $B_m$  with respect to powers of  $\chi$ .

(v). Take  $b \in B_m$ , then we have:

$$\langle b, 1 \rangle \sum_{l \geq 0} y_l = \langle b, 1 \rangle \lambda_B = b \lambda_B = b \sum_{l \geq 0} y_l.$$

Since  $by_0 \in B_m$  and the sum is direct, it follows that

$$(29) \quad by_0 = \langle b, 1 \rangle y_m.$$

Take  $b \in B_m$  so that  $\langle b, 1 \rangle \neq 0$ , (e.g.  $b = \chi^m$ ), then applying 1 to both sides of (29) yields

$$(30) \quad \dim B_0 = \dim B_m$$

for all  $0 \leq m \leq t-1$ . Since  $\chi$  is the only irreducible character in  $B_1$  it follows that  $\dim B_1 = \langle \chi, 1 \rangle^2$ , hence the result follows.  $\square$

As a corollary we obtain:

**Corollary 3.12.** With notations as in Proposition 3.11, we have:  
 $F_j b = \omega^m \langle b, 1 \rangle F_j$  for all  $b \in B_m$ .

*Proof.* Let  $b \in B_m$ . By (29) and (30) we have:

$$bF_j = b \frac{y_0}{\dim B_0} F_j = \frac{1}{\dim B_0} \langle b, 1 \rangle y_m F_j = \frac{1}{\dim B_0} \langle b, 1 \rangle \omega^m \dim B_m F_j = \langle b, 1 \rangle \omega^m F_j.$$

□

In what follows we wish to generalize an important normal subgroup that occurs in group theory to semisimple hopf algebras. Let  $G$  be a finite group and  $V$  a representation of  $G$  with associated representation  $\chi = \chi_V$ . Define

$$Z(\chi) = \{g \in G \mid |\langle \chi, g \rangle| = \langle \chi, 1 \rangle\}.$$

It is shown (see e.g. [I]) that

$$g \in Z(\chi) \Leftrightarrow g \cdot v = \omega_g v, \forall v \in V.$$

In this sense  $Z(\chi)$  is a ‘semi kernel’ of  $\rho_V$ . Based on this we define the  **$\omega$ -left kernel** as follows. Let  $V$  be a representation of  $H$ , set:

$$(31) \quad \text{LKer}_V^\omega = \{h \in H \mid h_1 \otimes h_2 \cdot v = \omega h \otimes v \quad \forall v \in V, |\omega| = 1\}$$

We show:

**Lemma 3.13.** *Let  $V$  be a representation of  $H$ .*

(i) *If  $0 \neq \text{LKer}_V^\omega$  then it is a non-zero left  $D(H)$ -module, that is, it is a left coideal stable under the left adjoint action of  $H$ .*

(ii)  $(\text{LKer}_V^\omega)(\text{LKer}_V^{\omega'}) \subset \text{LKer}_V^{\omega\omega'}$ .

(iii) *If  $\omega^l = 1$  then  $\text{LKer}_V^\omega \subset \text{LKer}_{V^{\otimes l}}^\omega$ .*

(iv) *Set  $Lz_\chi = \bigoplus_\omega \text{LKer}_V^\omega$ . Then  $Lz_\chi$  is a normal left coideal subalgebra of  $H$ .*

*Proof.* (i) The proof that  $\text{LKer}_V^\omega$  is a left coideal stable under the left adjoint action of  $H$  is exactly as for these properties for  $\text{LKer}_V$  as shown in [2.1.3][Bu2].

(ii) This follows by a direct computation.

(iii) Observe that  $\text{LKer}_{V_i}^\omega$  is a left coideal so that each  $h \in \text{LKer}_{V_i}^\omega$  acts on  $V_i$  as  $\omega \langle \varepsilon, h \rangle \text{Id}$ . Hence we have for all  $h \in \text{LKer}_{V_i}^\omega$ ,  $v_1, \dots, v_l \in V$ ,

$$h \cdot (v_1 \otimes \dots \otimes v_l) = \sum h_1 \cdot v_1 \otimes \dots \otimes h_l \cdot v_l = \omega^l \langle \varepsilon, h \rangle (v_1 \otimes \dots \otimes v_l).$$

By Remark 2.8(iii),  $\text{LKer}_V^\omega \subset \text{LKer}_{V^{\otimes l}}^\omega$ .

(iv) Follows from parts (i) and (ii). □

Since  $\omega$ -left kernels are left  $D(H)$ -modules, and since conjugacy classes are the irreducible left  $D(H)$ -modules it follows that  $\omega$ -left kernels are sums of conjugacy classes. In what follows we discuss further how they are related.

The following is the essential step in obtaining left kernels from the character table, generalizing the process of recovering normal subgroups of a finite group  $G$  from its character table. Denote by  $\mathfrak{C}^j$  the following left coideal of  $H$  which is stable under the left adjoint action:

$$\mathfrak{C}^j = \bigoplus_{k=1}^{m_j} \mathfrak{C}_{kj} = \Lambda \leftarrow F_j H^*.$$

Note that if  $H$  is almost cocommutative then  $\mathfrak{C}^j = \mathfrak{C}_j$ , where  $\mathfrak{C}_j$  is as defined in (4). Also note that

$$\eta_j \in \mathfrak{C}^j \quad \text{and} \quad \eta_j \leftarrow H^* = \mathfrak{C}^j.$$

We can prove now our main theorem.

**Theorem 3.14.** *Let  $H$  be a semisimple Hopf algebra over an algebraically closed field  $k$  of characteristic 0,  $\chi_i$  an irreducible character and  $F_j$  a central primitive idempotent of  $R(H)$ . Then the following are equivalent:*

- (i)  $F_j \chi_i = \omega_i \langle \chi_i, 1 \rangle F_j$ ,  $|\omega_i| = 1$ .
- (ii)  $\langle \chi_i, \eta_j \rangle = \omega_i m_j \langle \chi_i, 1 \rangle$ ,  $|\omega_i| = 1$ .
- (iii)  $\mathfrak{C}^j \subset \text{LKer}_{V_i}^{\omega_i}$ .

Moreover, if the equivalent conditions hold then  $\omega_i$  is a root of unity of order  $t$ , where  $t$  divides  $\dim H$ .

*Proof.* (i)  $\Leftrightarrow$  (ii) is Theorem 3.10(i).

(iii)  $\Rightarrow$  (ii). If  $\mathfrak{C}^j \subset \text{LKer}_{V_i}^{\omega_i}$  then in particular  $\eta_j \in \mathfrak{C}^j$  acts as  $\omega_i m_j$  on  $V_i$ , hence (ii) follows.

(i)  $\Rightarrow$  (iii). By Remark 2.8(iii) and since  $\mathfrak{C}^j$  is a left coideal, it is enough to show, that  $a \cdot v = \omega_i \langle \varepsilon, a \rangle v$  for all  $a \in \mathfrak{C}^j$ ,  $v \in V_i$ .

As a left  $H$ -module,  $V_i$  is a right  $H^*$ -comodule, with  $\rho(v) = \sum v_0 \otimes v_1$  for all  $v \in V_i$ . Take  $\chi = \chi_i$  in Proposition 3.11, then  $sp_{\mathbf{t}}\{v_1\} \subset B_1$  in the same proposition.

Let  $a = (\Lambda \leftarrow F_j p) \in \mathfrak{E}^j$ ,  $v \in V$ . Then:

$$\begin{aligned}
a \cdot v &= \\
&= \sum_v \langle \Lambda \leftarrow F_j p, v_1 \rangle v_0 \\
&= \sum_v \langle \Lambda, v_1 F_j p \rangle v_0 \quad (\text{since } \Lambda \text{ is cocommutative}) \\
&= \sum_v \langle v_1, 1 \rangle \omega_i \langle \Lambda, F_j p \rangle v_0 \quad (\text{by Corollary 3.12}) \\
&= \omega_i \langle \varepsilon, a \rangle v.
\end{aligned}$$

The fact that  $\omega$  is a root of unity follows from Proposition 3.10(iii). By Proposition 3.11(iv)-(v),  $(\dim B) = t(\dim B_0)$ . Since  $(\dim B) | (\dim H)$  the last result follows.  $\square$

As a corollary we obtain:

**Theorem 3.15.** *Let  $H$  be a semisimple Hopf algebra over  $\mathbb{C}$ ,  $B$  the Hopf subalgebra of  $H^*$  generated by  $\chi_i$ , and  $F_j$  a central primitive idempotent of  $R(H)$ . Then the following are equivalent:*

- (i)  $F_j \chi_i = \langle \chi_i, 1 \rangle F_j$
- (ii)  $\langle \chi_i, \eta_j \rangle = m_j \langle \chi_i, 1 \rangle$ .
- (iii)  $\mathfrak{E}^j \subset \text{LKer}_{V_i} = H^B$ .

In the next section (Lemma 4.3), we show that  $\omega$ -left kernels are abundant for factorizable Hopf algebras.

#### 4. APPLICATIONS

In this section we prove properties of semisimple Hopf algebras over  $\mathbb{C}$  related to their character table with particular emphasis on Burnside's  $p^a q^b$  theorem. It is known that this theorem fails if we use a sequence of normal Hopf subalgebras to define solvability. (A counter example of dimension 36 is given in [GN]).

A categorical version of Burnside's theorem is given in [ENO]. They prove that any fusion category of Frobenius-Perron dimension  $p^a q^b$  is solvable. However, it is not clear how to interpret all the categorical notions to Hopf algebras. The discussion in this section indicates that a sequence of normal left coideal subalgebras should be used to define solvability in this case. We prove some of the results in [ENO] for semisimple Hopf algebras of dimension  $p^a q^b$  by using the algebraic tools we developed in the previous sections.

We start by proving a Hopf algebra analogue of the following:



**A theorem of Burnside**(see e.g. [I, Th. 3.8]): If  $G$  is a non-abelian simple group then  $\{1\}$  is the only conjugacy class of  $G$  which has prime power order.

Note that 1 is considered as a prime power as well. The discussion goes further and we prove some structural properties mainly of factorizable Hopf algebras.

Recall, if  $(H, R)$  is a finite dimensional quasitriangular Hopf algebra then the maps  $f_R : H^{*cop} \rightarrow H$ , defined by  $f_R(p) = \langle p, R^1 \rangle R^2$  and  $f_R^* : H^{*op} \rightarrow H$ , defined by  $f_R^*(p) = \langle p, R^2 \rangle R^1$  are Hopf algebra maps. Then  $Q = R^{21}R$  and the Drinfeld map  $f_Q$  is given by  $f_Q = f_R^* * f_R$ . When  $(H, R)$  is a quasitriangular Hopf algebra then  $R(H)$  is necessarily commutative. If  $H$  is also semisimple then the Drinfeld map  $f_Q$  is an algebra isomorphism between  $R(H)$  and  $Z(H)$ .

The  $S$ -matrix for  $(H, R)$  is defined by:

$$(32) \quad s_{ij} = \langle \chi_i, f_Q(\chi_j) \rangle.$$

The quasitriangular Hopf algebra  $(H, R)$  is *factorizable* if  $f_Q$  is a monomorphism, or equivalently, if its  $S$ -matrix is invertible.

Assume  $(H, R)$  is factorizable. Reorder the set  $\{F_j\}$  so that

$$f_Q(F_j) = E_j$$

for all  $1 \leq j \leq m$ . It follows that  $\dim(F_j H^*) = d_j^2$ , where  $d_j = \langle \chi_j, 1 \rangle$ . Recall [CW2] that, after reordering the central primitive idempotents, we have:

$$(33) \quad f_Q(\chi_j) = \frac{1}{d_j} C_j = d_j \eta_j.$$

The last equality follows since  $\dim(F_j H^*) = \dim(E_j H) = d_j^2$ . It follows that the  $S$ -matrix satisfies

$$(34) \quad s_{ij} = d_j \xi_{ij}.$$

In what follows we discuss further properties of  $f_Q$ .

**Lemma 4.1.** *Let  $(H, R)$  be a quasitriangular semisimple Hopf algebra over  $k$ . Then  $f_Q$  maps coalgebras of  $H^*$  to left coideals stable under the adjoint action of  $H$  and Hopf subalgebras of  $H^*$  to normal left coideal subalgebras of  $H$ .*

*Proof.* Let  $C$  be a subcoalgebra of  $H^*$ ,  $c \in C$ . By [CW1, Prop.2.5(3)],

$$f_Q(c) \leftarrow p = f_Q(f_R^*(p_2) \rightarrow c \leftarrow f_R(p_1)) \in f_Q(C),$$

for all  $p \in H^*$ . Since  $C$  is a coalgebra we have that  $f_Q(C)$  is a left coideal of  $H$ . It is stable under the adjoint action of  $H$  by [CW1, Prop.2.5(5)].

If  $C$  is a Hopf subalgebra of  $H^*$  then all that is left to show is that  $f_Q(C)$  is an algebra. Let  $x, y \in C$ , then by the definition of  $f_Q$  we have,

$$f_Q(x)f_Q(y) = f_R^*(x_1)f_R(x_2)f_Q(y) = f_R^*(x_1)[f_R(x_3)_{ad}f_Q(y)]f_R(x_2),$$

where the last equation follows since  $f_R$  is an anticoalgebra map. Since  $f_Q(C)$  is stable under the adjoint action of  $H$ , the mid term above belongs to  $f_Q(C)$ . The result follows now from [CW1, Prop.2.5(2)].  $\square$

If  $(H, R)$  is factorizable more can be said:

**Lemma 4.2.** *Let  $(H, R)$  be a factorizable semisimple Hopf algebra over  $k$ . Then,*

(i) *For all  $0 \leq j \leq n-1$ , let  $R_{V_j}$  be the coalgebra generated by  $\chi_j$ , then  $f_Q(R_{V_j}) = \mathfrak{C}_j$ .*

(ii) *Let  $N$  be a left coideal of  $H$ . If a character  $\chi \in f_Q^{-1}(N)$  then the same hold for all its constituents.*

(iii) *Let  $N$  be a normal left coideal subalgebra of  $H$ . Then  $f_Q^{-1}(N)$  is a Hopf subalgebra of  $H^*$ .*

*Proof.* (i) By Lemma 4.1,  $f_Q(R_{V_j})$  is a left coideal of  $H$ . By (33),  $\eta_j \in f_Q(R_{V_j})$ . Since  $\mathfrak{C}_j$  is the left coideal generated by  $\eta_j$  we have  $\mathfrak{C}_j \subset f_Q(R_{V_j})$ . But  $f_Q$  is injective and  $\dim \mathfrak{C}_j = d_j^2 = \dim R_{V_j}$ , hence equality holds.

(ii) Let  $\chi = \sum_{j \in J} m_j \chi_j$ ,  $m_j \in \mathbb{Z}^+$ . By (33),  $f_Q(\chi) = \sum_{j \in J} d_j m_j \eta_j \in N$ . For each  $j \in J$ , we have  $d_j m_j \eta_j = f_Q(\chi) \leftarrow F_j \in N$ . Thus  $\chi_j \in f_Q^{-1}(N)$ .

(iii) Since  $N$  is a normal left coideal subalgebra, it is a left  $D(H)$ -module, hence by (5), there exists a set  $J \subset \{0, \dots, n-1\}$  so that  $N = \oplus_{j \in J} \mathfrak{C}_j$ . By (i)  $f_Q^{-1}(N) = \oplus_{j \in J} (R_{V_j})$  hence it is a coalgebra. Let  $\chi_k, \chi_l \in f_Q^{-1}(N)$ . Since  $f_Q$  is multiplicative on  $R(H)$  and  $N$  is an algebra we have:

$$f_Q(\chi_k \chi_l) = f_Q(\chi_k) f_Q(\chi_l) \in N.$$

Hence  $\chi_k \chi_l \in f_Q^{-1}(N)$ . By (ii) each constituent of  $\chi_k \chi_l \in f_Q^{-1}(N)$  as well.

We want to show that  $f_Q^{-1}(N)$  is an algebra. Let  $a \in R_{V_j}$ ,  $b \in R_{V_k}$ , where  $R_{V_j}, R_{V_k} \subset f_Q^{-1}(N)$ . In particular  $\chi_j, \chi_k \in N$ . Assume  $\chi_j \chi_k = \sum m_i \chi_i$ , then by (ii), each  $\chi_i \in f_Q^{-1}(N)$  and thus by (i), each  $R_{V_i} \subset N$ . We have now,

$$ab \in R_{V_j} R_{V_k} = R_{V_j \otimes V_k} = \oplus R_{V_i} \subset f_Q^{-1}(N).$$

$\square$

We show now how  $\omega$ -left kernels arise for factorizable Hopf algebras.

**Lemma 4.3.** *Let  $(H, R)$  be a factorizable semisimple Hopf algebra over  $\mathbb{C}$ , then*

(i)  $\frac{d_j \langle \chi_i, \eta_j \rangle}{d_i}$  is an algebraic integer for all  $i, j$ .

(ii) If  $\gcd(d_i, d_j) = 1$  and  $\langle \chi_i, \eta_j \rangle \neq 0$ , then  $\omega_i = \frac{\langle \chi_i, \eta_j \rangle}{d_i}$  is a root of 1. Thus  $\mathfrak{C}^j \subset \text{LKer}_{V_i}^{\omega_i}$ .

*Proof.* (i) Since  $\frac{s_{ij}}{d_i}$  is known to be an algebraic integer, the result follows from (34).

(ii) Assume  $\langle \chi_i, \eta_j \rangle \neq 0$ . Since  $\gcd(d_i, d_j) = 1$ , we may choose integers  $u, v$  so that  $ud_i + vd_j = 1$ . Multiplying both sides by  $\frac{\langle \chi_i, \eta_j \rangle}{d_i}$ , we get

$$\frac{\langle \chi_i, \eta_j \rangle}{d_i} = u \langle \chi_i, \eta_j \rangle + v \frac{\langle \chi_i, \eta_j \rangle d_j}{d_i}$$

The first summand on the right hand side is an algebraic integer, which we show to be a root of unity. by Corollary 3.3, while the second one is by the first part. It follows that  $\frac{\langle \chi_i, \eta_j \rangle}{d_i}$  is an algebraic integer.

Recall, [V], that  $Q$  has finite order, hence its action on  $V_i \otimes V_j$  is diagonalizable with eigenvalues which are roots of unity. Since  $\chi_i \otimes \chi_j$  is the trace of  $H \otimes H$  acting on  $V_i \otimes V_j$  it follows that

$$\langle \chi_i \otimes \chi_j, Q \rangle = \sum_{k=1}^{d_i d_j} \omega_k,$$

where  $\omega_k$  is a root of unity. Since the  $S$ -matrix is symmetric we have by (32) and (34),

$$\frac{\langle \chi_i, \eta_j \rangle}{d_i} = \frac{s_{ji}}{d_i d_j} = \frac{\langle \chi_i \otimes \chi_j, Q \rangle}{d_i d_j} = \frac{1}{d_i d_j} \sum_{k=1}^{d_i d_j} \omega_k$$

Since  $\frac{\langle \chi_i, \eta_j \rangle}{d_i}$  is an algebraic integer, which is nonzero by assumption, it follows by elementary Galois theory (see e.g [I, 3.8]), that all  $w_k$  are equal and thus the left handside is a root of unity.

The last part follows since factorizable Hopf algebras are almost co-commutative, hence we have  $m_j = 1$  and  $\mathfrak{C}_j = \mathfrak{C}^j$ , the result follows from Theorem 3.15.  $\square$

The following lemma arises from the proof of Burnside  $p^a q^b$  theorem for groups.

**Lemma 4.4.** *Let  $H$  be a semisimple Hopf algebra over  $\mathbb{C}$ ,  $q$  a prime number. Then for each  $0 < j \leq m-1$  there exists  $0 < i \leq n-1$  so that  $\langle \chi_i, \eta_j \rangle \neq 0$  and  $q$  does not divide  $d_i$ .*

*Proof.* By the orthogonality relations with respect to the first column, (Theorem 3.5), we have,

$$(35) \quad 0 = \sum_m d_m \langle \chi_m, \eta_j \rangle$$

We claim that there exists  $i > 0$  so that  $q$  does not divide  $d_i$  and  $\langle \chi_i, \eta_j \rangle \neq 0$ . Indeed, let  $S$  be the set of all  $V_m$  so that  $q|d_m$  and let  $T$  be the set of all non-trivial  $V_m$  so that  $q$  does not divide  $d_m$ . By (35),

$$0 = 1 + \sum_{V \in S} d_m \langle \chi_m, \eta_j \rangle + \sum_{V \in T} d_m \langle \chi_m, \eta_j \rangle$$

Set

$$a = \sum_{V \in S} \frac{d_m}{q} \langle \chi_m, \eta_j \rangle$$

Then  $a$  is an algebraic integer and

$$0 = 1 + qa + \sum_{V \in T} d_m \langle \chi_m, \eta_j \rangle.$$

If the last summand is 0, then  $a$  is a rational algebraic integer, hence an integer which is impossible. Hence there exists  $i$  so that  $\langle \chi_i, \eta_j \rangle \neq 0$  and  $q \nmid d_i$ .  $\square$

We are ready to prove a Hopf algebra analogue the theorem of Burnside for groups mentioned at the beginning of this section.

**Theorem 4.5.** *Let  $(H, R)$  be a non-commutative quasitriangular semisimple Hopf algebra over  $\mathbb{C}$  whose only normal left coideal subalgebras are  $\mathbb{C}$  and  $H$ . Then  $\mathbb{C}$  is the only conjugacy class of  $H$  with a prime power dimension.*

*Proof.* By Lemma 4.1,  $\text{Im} f_Q$  is a normal left coideal sbalgebra of  $H$ , hence by assumption either (i)  $\text{Im} f_Q = \mathbb{C}$  or (ii)  $\text{Im} f_Q = H$ .

(i) If  $\text{Im} f_Q = \mathbb{C}$  then  $Q = 1 \otimes 1$  hence  $H$  is triangular. In this case the Drinfeld element  $u$ , is a central grouplike element of order  $\leq 2$  [Dr, Ra1].

If  $u \neq 1$  then  $k\langle u \rangle$  is a normal left coideal subalgebra hence equals  $H$ , contradicting non-commutativity.

If  $u = 1$  then by [EG],  $H = (kG)^J$  where  $J$  is a twist for  $kG$ . By [CW2, Cor.2.12]  $H$  has the same character table as  $kG$  and thus has the same lattice of normal left coideal subalgebras and by Remark 3.6,

the same dimensions of conjugacy classes. By the theorem of Burnside for groups,  $G$  has no conjugacy class with a prime power order, hence neither does  $H$ .

(ii) If  $\text{Im} f_Q = H$  then  $H$  is factorizable. Let  $\mathcal{C}_j$ ,  $j > 0$ , be any conjugacy class. If  $\dim \mathcal{C}_j = 1$  then  $\eta_j$  is a central grouplike element of  $H$  implying that the group algebra generated by  $\eta_j$  is a normal left coideal subalgebra of  $H$ . Non-commutativity of  $H$  and the hypothesis of the theorem imply  $\eta_j = \varepsilon$ , a contradiction to  $j > 0$ .

Assume  $\dim \mathcal{C}_j = q^l \neq 1$ . By Lemma 4.4 there exists  $i > 0$  so that  $V_i$  has dimension prime to  $q$  and  $\langle \chi_i, \eta_j \rangle \neq 0$ . Since  $H$  is factorizable we have  $\dim(\mathfrak{C}_j) = d_j^2$ , and so  $(d_i, d_j) = 1$ . By Lemma 4.3(ii), there exists  $\omega$ ,  $\omega^l = 1$  and  $\mathfrak{C}_j \subset \text{LKer}_{V_i}^\omega$ , which by Lemma 3.13(iii) is contained in  $\text{LKer}_{V_i^{\otimes l}}$ . In particular  $\mathfrak{C} \neq \text{LKer}_{V_i^{\otimes l}}$ .

Thus if  $d_i \neq 1$  we have  $\chi_{V_i^{\otimes l}} \neq \varepsilon$  hence by Remark 2.8(ii),  $\text{LKer}_{V_i^{\otimes l}} \subsetneq H$ , a contradiction. If  $d_i = 1$  then by Theorem 2.13,  $H$  must be commutative, a contradiction.  $\square$

The following results, culminating in Theorem 4.9, follow directly from [ENO] who obtained it using categorical considerations. We represent it here in algebraic terms as consequences of the previous discussion.

**Proposition 4.6.** *Let  $H$  be a quasitriangular semisimple Hopf algebra over  $\mathbb{C}$  of dimension  $p^a q^b$ ,  $a + b > 1$ ,  $p, q$  prime numbers. Then  $H$  has a proper non-trivial normal left coideal subalgebra.*

*Proof.* If  $H$  is commutative then quasitriangularity implies that it is cocommutative, hence a group algebra. Since the order of the group is not prime, it has a normal subgroup. Hence  $H = kG$  has a non-trivial normal left coideal subalgebra.

Assume  $H$  is non-commutative and  $a > 0$ . We claim that there exists  $j > 0$  so that  $p$  does not divide  $\dim \mathfrak{C}_j$ . This follows from the fact that the dimension of each conjugacy class divides the dimension of  $H$  ([Kac, Z]), and their sum equals  $\dim H$ . Since  $\mathfrak{C}_0 = \mathbb{C}$ ,  $p$  can not divide the dimensions of all other conjugacy classes. That is, there exists  $j > 0$  so that  $\mathcal{C}_j$  has a prime power dimension. By Theorem 4.5,  $H$  contains a proper non-trivial normal left coideal subalgebra.  $\square$

**Corollary 4.7.** *Let  $H$  be a semisimple quasitriangular Hopf algebra of dimension  $p^a q^b$ ,  $a + b > 0$ , over  $\mathbb{C}$ . Then  $H$  admits a nontrivial 1-dimensional representation.*

*Proof.* Let  $M \neq H$  be a maximal normal left coideal subalgebra of  $H$ . Then  $\overline{H} = H/(M^+H)$  has no non-trivial proper normal left coideal subalgebras and  $\dim \overline{H} = p^{a'}q^{b'}$ ,  $a' + b' > 0$ . Proposition 4.6 implies now that  $a' + b' = 1$ . Hence  $\overline{H} \cong \mathbb{Z}_p$  and thus has a 1-dimensional representation which is also an  $H$ -representation.  $\square$

We can show now:

**Proposition 4.8.** [ENO, Prop.9.9]. *Let  $H$  be a semisimple Hopf algebra over  $\mathbb{C}$  of dimension  $p^aq^b$ ,  $a + b > 0$ ,  $p, q$  prime numbers. Then  $H$  admits a non-trivial 1-dimensional representation and a grouplike element  $\sigma \neq 1$ .*

*Proof.* If  $a + b = 1$  then  $H$  is a group algebra of prime order and we are done. Otherwise, consider the factorizable Hopf algebra  $D(H)$ . By Corollary 4.7 it has a non-trivial 1-dimensional representation, hence  $D(H)^*$  has a non-trivial grouplike element, which by [Ra2], has the form  $\sigma \otimes \tau$ , where  $\sigma \in G(H)$ ,  $\tau \in G(H^*)$ . If  $\tau \neq \varepsilon$  and  $\sigma \neq 1$  then we are done.

Assume now  $\tau = \varepsilon$ . By [Dr, Ra2],  $\sigma \neq 1$  is a central grouplike element of  $H$ . Let  $N$  be the central Hopf subalgebra of  $H$  generated by  $\sigma$ . Then by the induction hypothesis  $H/HN^+$  has a 1-dimensional representation hence so does  $H$ . The same proof, replacing  $H$  with  $H^*$ , works when  $\sigma = 1$ .  $\square$

These culminate in:

**Theorem 4.9.** *Let  $H$  be a semisimple Hopf algebra over  $\mathbb{C}$  of dimension  $p^aq^b$ ,  $a + b > 0$ ,  $p, q$  prime numbers. Then either  $H$  is a group algebra of prime order, or  $H$  contains a normal left coideal subalgebra  $N$  so that  $\mathbb{C} \subsetneq N \subsetneq H$ .*

*Proof.* By Proposition 4.8,  $H$  has a non-trivial 1-dimensional representation  $V$ , hence  $\text{LKer}_V \neq H$  is a normal left coideal subalgebra of  $H$ . If  $V$  is not faithful then  $\mathbb{C} \subsetneq \text{LKer}_V$  and we are done. Otherwise, by Theorem 2.13,  $H$  is a group algebra. If  $a + b > 1$  then this group contains a normal subgroup  $N$ , hence contains  $kN$ , a normal left coideal subalgebra. If  $a + b = 1$  then  $H$  is a group algebra of prime order.  $\square$

When  $H$  is factorizable more can be said:

**Theorem 4.10.** *Let  $H$  is a factorizable semisimple Hopf algebra of dimension  $p^aq^b$ ,  $a + b > 0$ , and let  $N$  be a normal left coideal subalgebra of  $H$ . Then  $N$  contains a central grouplike element. In particular, any minimal normal left coideal subalgebra of  $H$  is a central Hopf subalgebra of  $H$  of prime order.*

*Proof.* By Lemma 4.2 and Proposition 4.8, the Hopf algebra  $f_Q^{-1}(N)$  contains a non-trivial grouplike element. Since  $f_Q$  maps grouplike elements of  $H^*$  to central grouplike elements of  $H$  we are done. The rest is straightforward.  $\square$

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